Supersymmetry for integrable hierarchies on loop superalgebras

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Abstract

The algebraic approach is employed to formulate $N = 2$ supersymmetry transformations in the context of integrable systems based on loop superalgebras $\widehat{\mathrm{sl}}(p+1, p)$, $p \geq 1$ with homogeneous gradation. We work with extended integrable hierarchies, which contain supersymmetric AKNS and Lund-Regge sectors.

We derive the one-soliton solution for $p = 1$ which solves positive and negative evolution equations of the $N = 2$ supersymmetric model.

1 Introduction

The well-known examples of the mKdV and sine-Gordon models [1] as well as the AKNS and the Lund-Regge (Complex sine-Gordon) models [2] can be understood as positive and negative flows belonging to the same integrable hierarchy conveniently classified in terms of underlying algebraic structure. The basic objects in an algebraic approach to integrable models are the loop algebra $\widehat{\mathcal{G}}$ endowed with a grading operator Q and a constant semisimple generator E. The grading decomposes the loop algebra into graded subspaces and specifies the space of physical fields.

There already exists an extensive literature devoted to construction of integrable models with extended supersymmetry based on superspace formalism, (see e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]). Here, purely algebraic approach without superspace techniques is employed to construct a class of integrable hierarchies based on loop superalgebras $\widehat{\mathrm{sl}}(p+1, p)$, $p \geq 1$ with homogeneous gradation. Within this framework the symmetry structure of the underlying model is realized as a graded subalgebra of $\widehat{\mathrm{sl}}(p+1, p)$ defined as a centralizer $\mathcal{K}$ of a semisimple element E ($\mathcal{K} = \mathrm{Ker}(E)$). The positive and negative grade elements in the center of $\mathcal{K}$ generate positive and negative evolution flows of the extended hierarchy. The

positive flows are of the generalized AKNS type, while the first negative flow corresponds to the Lund-Regge (relativistic) model. By comparing expressions for subsequent positive flows we derive the closed expression for the recursion operator $\mathcal{R}$.

The elements in $\mathcal{K}$ not residing in the center give rise to the non-abelian symmetry structure of the model. In particular, $N = 2$ supersymmetry transformations arise in this context being generated by specific fermionic generators $f_\pm \in \mathcal{K}$. The supersymmetry algebra follows from the fact that the anti-commutator of $f_\pm$ closes on E [3, 4]

Furthermore, the algebraic approach allows a systematic construction of Hamiltonians. We derive the Lagrangian density for the second flow and the canonical (Dirac) bracket which defines the first Poisson bracket structure of the model. Higher brackets can be obtained by successive applications of recursion operator.

We employ the dressing and vertex formalisms to construct soliton solutions. Explicit formulas for the one-soliton solution are found for $p = 1$. The formula is verified to be a solution of both the AKNS and Lund-Regge type of evolution equations.

This work completes the discussion started in [15], where integrable models based on loop superalgebras with principal gradation (like supersymmetric KdV model) were treated by algebraic methods.

Section 2, gives details of algebraic foundation of our construction. We write equations of motion governed by the second flow for the $\widehat{\mathrm{sl}}(p+1,p)$ model for arbitrary p generalizing the results of [14]. Symmetries, conservation laws, recursion operator and the first Poisson bracket are presented within the algebraic framework in section 3. In section 4, we deal with the first negative flow of the extended hierarchy, which is shown to be invariant under $N = 2$ supersymmetry. Next we write explicit the Lund-Regge equations of motion (for the $p = 1$) model using the Gauss decomposition of the zero grade subgroup element B. In section 5, we explicit construct one soliton solution by using the dressing and vertex operator techniques.

2 General Construction of the Model

Consider the graded Lie algebra $\mathrm{sl}(p+1, s+1)$ composed of bosonic (even) α_i, β_t, $i = 1, \dots, p$, $t = 1, \dots, s$ and fermionic (odd), γ, simple roots:

$$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \quad i = 1, \dots, p \\ \beta_t &= f_t - f_{t+1}, \quad t = 1, \dots, s \\ \gamma &= e_{p+1} - f_1 \end{aligned} \tag{1}$$

in terms of unit length vectors e_r, f_a such that $e_r^2 = 1$ and $f_a^2 = -1$ with $r = 1, \dots, p+1$ and $a = 1, \dots, s+1$. The remaining bosonic roots can be written as

$$\pm(\alpha_r + \cdots + \alpha_{q-1}) = \pm(e_r - e_q), \quad \pm(\beta_a + \cdots + \beta_{b-1}) = \pm(f_a - f_b), \tag{2}$$

where $r, q = 1, \cdots, p+1$ and $a, b = 1, \cdots, s+1$. The remaining fermionic roots have the form

$$\pm(\alpha_r + \cdots + \alpha_{p+1} + \gamma + \beta_1 + \cdots + \beta_{a-1}) = \pm(e_r - f_a). \tag{3}$$

Consider the homogeneous gradation with the grading operator $Q = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}$. The affine super Lie algebra $\widehat{\mathcal{G}} = \widehat{\mathrm{sl}}(p+1, s+1)$ decomposes according to $\widehat{\mathcal{G}} = \oplus \mathcal{G}_k$ with the graded subspaces $\mathcal{G}_k$ that satisfy $[Q, \mathcal{G}_k] = k\mathcal{G}_k$ with $k \in \mathbb{Z}$ and consist of elements $X^{(k)} = \lambda^k X$ for $X \in \mathrm{sl}(p+1, s+1)$.

The key role in an algebraic approach to integrable models is played by the constant grade one generator

$$E = E^{(1)} \tag{4}$$

obtained by setting $n = 1$ in the following expression for the semisimple generators of grade n

$$E^{(n)} = \left(\sum_{i=1}^{p} e_i - \sum_{a=1}^{s+1} f_a \right) \cdot H^{(n)} .$$

The centralizer $\mathcal{K} = \mathrm{Ker}(E)$ is a subalgebra of $\mathcal{G}$ spanned by the Cartan subalgebra of $\mathrm{sl}(p+1, s+1)$ and the step operators associated to roots

$$\begin{aligned}
&\pm(\alpha_i + \cdots + \alpha_j) = \pm(e_i - e_j), \quad \pm(\beta_a + \cdots + \beta_{b-1}) = \pm(f_a - f_b), \\
&\pm(\alpha_i + \cdots \alpha_p + \gamma + \beta_1 + \cdots + \beta_{a-1}) = \pm(e_i - f_a), \\
&i, j = 1, \cdots, p, \quad a, b = 1, \cdots, s+1
\end{aligned} \tag{5}$$

The image $\mathcal{I} = \mathrm{Im}(E)$ is then obtained as a linear combination of step operators of bosonic roots

$$\pm(\alpha_i + \cdots \alpha_p) = \pm(e_i - e_{p+1}), \quad i = 1, \cdots, p$$

and fermionic roots

$$\pm(\gamma + \beta_1 + \cdots + \beta_{a-1}) = \pm(e_{p+1} - f_a), \quad a = 1, \cdots, s+1 ,$$

respectively.

The fact that the element E is semisimple insures that $\mathcal{G} = \mathcal{K} \oplus \mathcal{I}$. In addition, it also holds that $[\mathcal{I}, \mathcal{I}] \subset \mathcal{K}$.

Because supersymmetry requires equal number of bosons and fermions, we impose from now on the condition $p = s+1$ and denote $\alpha_{p+1} = \gamma$, $\alpha_{p+2} = \beta_1$, $\alpha_{p+3} = \beta_2, \cdots, \alpha_{2p} = \beta_{p-1}$.

The super algebra $\widehat{\mathrm{sl}}(p+1, p)$ gives rise to a supersymmetric integrable model specified by element E of eqn. (4) and the following Lax operator :

$$L = \partial_x + E + A_0 = \partial_x + \mathcal{A}_x , \tag{6}$$

where $\mathcal{A}_x = E + A_0$ and

$$\begin{aligned}
A_0 &= \sum_{i=1}^{p} \left(\bar{b}_i E^{(0)}_{e_i - e_{p+1}} + b_i E^{(0)}_{-(e_i - e_{p+1})} \right) + \sum_{a=1}^{p} \left(\bar{F}_a E^{(0)}_{f_a - e_{p+1}} + F_a E^{(0)}_{-(f_a - e_{p+1})} \right) , \\
&= \sum_{i=1}^{p} \left(\bar{b}_i E^{(0)}_{\alpha_i + \cdots + \alpha_p} + b_i E^{(0)}_{-(\alpha_i + \cdots + \alpha_p)} \right) + \sum_{a=1}^{p} \left(\bar{F}_a E^{(0)}_{-(\alpha_{p+1} + \cdots + \alpha_{p+a})} + F_a E^{(0)}_{\alpha_{p+1} + \cdots + \alpha_{p+a}} \right) .
\end{aligned} \tag{7}$$

lies in grade zero sector of $\mathcal{I}$.

Now, we propose the zero curvature relation $[\partial_{t_n} + \mathcal{A}_{t_n}, \partial_x + \mathcal{A}_x] = 0$. We search for solution of the form

$$\mathcal{A}_{t_n} = D^{(n)} + D^{(n-1)} + \cdots D^{(0)} \,, \tag{8}$$

where $D^{(k)} \in \mathcal{G}_k$. The zero curvature relation decomposes grade by grade into the following chain of equations,

$$\begin{aligned} [E, D^{(n)}] &= 0, \\ -\partial_x D^{(n)} + [A_0, D^{(n)}] + [E, D^{(n-1)}] &= 0, \\ \vdots \quad & \quad \vdots \\ -\partial_x D^{(1)} + [A_0, D^{(1)}] + [E, D^{(0)}] &= 0, \\ -\partial_x D^{(0)} + \partial_{t_2} A_0 + [A_0, D^{(0)}] &= 0 \,. \end{aligned} \tag{9}$$

The top equation in (9) implies that $D^{(n)} \in \mathcal{K}$ and, consequently, we assume that $D^{(n)} = E^{(n)}$.

Solving eqns. (9) from the highest to the zero grade components we find for $t_n = t_2$, the following equations of motion for the coefficients of A_0:

$$\begin{aligned} &\partial_{t_2} b_i + \partial_x^2 b_i - 2\sum_{j=1}^{p} \left(b_j \bar{b}_j + F_j \bar{F}_j\right) b_i = 0, \\ &\partial_{t_2} \bar{b}_i - \partial_x^2 \bar{b}_i + 2\sum_{j=1}^{p} \left(b_j \bar{b}_j + F_j \bar{F}_j\right) \bar{b}_i = 0, \\ &\partial_{t_2} F_i + \partial_x^2 F_i - 2\sum_{j=1}^{p} \left(b_j \bar{b}_j + F_j \bar{F}_j\right) F_i = 0, \\ &\partial_{t_2} \bar{F}_i - \partial_x^2 \bar{F}_i + 2\sum_{j=1}^{p} \left(b_j \bar{b}_j + F_j \bar{F}_j\right) \bar{F}_i = 0, \end{aligned} \tag{10}$$

where $i = 1, \cdots, p$. These result generalizes equations of motion found in [13] in the $p = 1$ case.

3 Symmetries and Conservation Laws

The symmetries of integrable models will be defined below in a systematic manner in the setting of a gauge transformation relating the bare Lax $(\partial_x + E)$ to the dressed Lax operator L given by definition (6) [17].

The dressing operator is defined by

$$\Theta = e^{\theta^{(-1)} + \theta^{(-2)} + \cdots}, \qquad \theta^{(-k)} \in \mathcal{G}_{-k}$$

and satisfies the relation:

$$E = \Theta^{-1} \left(E + A_0\right) \Theta + \Theta^{-1} \partial_x \Theta \,. \tag{11}$$

Decomposing (11) according to the grading we find the following equations,

$$
\begin{aligned}
A_0 + [E, \theta^{(-1)}] &= 0, \\
\partial_x \theta^{(-1)} + [E, \theta^{(-2)}] + \frac{1}{2}[A_0, \theta^{(-1)}] &= 0, \\
\partial_x \theta^{(-2)} + [E^{(1)}, \theta^{(-3)}] + \frac{1}{2}[A_0, \theta^{(-2)}] + \frac{1}{12}[[A_0, \theta^{(-1)}], \theta^{(-1)}] &= 0, \\
\partial_x \theta^{(-3)} + [E^{(1)}, \theta^{(-4)}] + \frac{1}{2}[A_0, , \theta^{(-3)}] + \frac{1}{12}[[A_0, \theta^{(-1)}], \theta^{(-2)}]& \\
+\frac{1}{12}[[A_0, \theta^{(-2)}], \theta^{(-1)}]] &= 0 \\
&\vdots
\end{aligned}
\tag{12}
$$

Decompose $\theta^{(-i)}$ in its components in the centralizer $\mathcal{K}$ and image $\mathcal{I}$ according to $\theta^{(-i)} = \theta^{(-i)}_{\mathcal{K}} + \theta^{(-i)}_{\mathcal{I}}$. Then the top eqn. (12) yields

$$
\theta^{(-1)}_{\mathcal{I}} = \sum_{i=1}^{p} \left(-\bar{b}_i E^{(0)}_{\alpha_i + \cdots + \alpha_p} + b_i E^{(0)}_{-(\alpha_i + \cdots + \alpha_p)} - \bar{F}_a E^{(0)}_{-(\alpha_{p+1} + \cdots + \alpha_{p+i})} + F_a E^{(0)}_{\alpha_{p+1} + \cdots + \alpha_{p+i}} \right), \tag{13}
$$

The second eqn. (12) can be rewritten as

$$
\begin{aligned}
\partial_x \theta^{(-1)}_{\mathcal{K}} + \frac{1}{2}[A_0, \theta^{(-1)}_{\mathcal{I}}] &= 0, \\
\partial_x \theta^{(-1)}_{\mathcal{I}} + [E, \theta^{(-2)}_{\mathcal{I}}] + \frac{1}{2}[A_0, \theta^{(-1)}_{\mathcal{K}}] &= 0
\end{aligned}
\tag{14}
$$

and can be used to determine a local expression for $\theta^{(-2)}_{\mathcal{I}}$.

An element $X_m \in \mathrm{Ker}(E)$ of m-th grade generates a symmetry transformation via [17]:

$$
\delta_{X_m} \Theta = \left(\Theta X_m \Theta^{-1} \right)_- \Theta \,. \tag{15}
$$

We will derive the corresponding transformation of A_0 using that from the top relation in (12) it follows that

$$
A_0 = [\theta^{(-1)}, E] = \left([\Theta, E] \right)_0 \,, \tag{16}
$$

where the right hand side contains projection on the zero grade.

Employing the definition (15) of the symmetry transformations to the above relation we find:

$$
\begin{aligned}
\delta_{X_m} A_0 &= \left([\left(\Theta X_m \Theta^{-1} \right)_{-1} \Theta, E] \right)_0 = [\left(\Theta X_m \Theta^{-1} \right)_{-1}, E] \\
&= \left([\Theta X_m \Theta^{-1}, E] \right)_0 = \left(\left[[\Theta, E] \Theta^{-1} \,,\, \Theta X_m \Theta^{-1} \right] \right)_0 \,.
\end{aligned}
$$

From equation (11) we find that

$$
[\Theta, E]\Theta^{-1} = A_0 + \partial_x \Theta \, \Theta^{-1}
$$

and, thus, transformation of A_0 can be given by a general gauge transformation formula:

$$\delta_{X_n} A_0 = \left(\left[A_0 + \partial_x \Theta\,\Theta^{-1}\,,\, \Theta X_n \Theta^{-1}\right]\right)_0 \,, \tag{17}$$

where X_n is an element in $\mathcal{K}$ of n-th grade. For $m = 0$, we find

$$\delta_{X_0} A_0 = [A_0\,,\, X_0]\,. \tag{18}$$

For $m = 1$, eqn. (17) yields,

$$\delta_{X_1} A_0 = [A_0, [\theta^{(-1)}, X_1]] + [\partial_x \theta^{(-1)}, X_1]\,. \tag{19}$$

In particular, for $X_1 = E$ we find that $\delta_E A_0 = \partial_x A_0 = \partial A_0/\partial t_1$, in accordance with the fact that the center of $\mathcal{K}$ generates isospectral flows.

Define two odd elements of $\mathrm{Ker}(E)$:

$$f_\pm = \sum_{i=1}^{p} \left(\epsilon^{(0)} E^{(0)}_{e_i - f_i} \pm \epsilon^{(1)} E^{(1)}_{-(e_i - f_i)}\right) \,, \tag{20}$$

containing grade 0 and 1 generators and Grassmannian parameters $\epsilon^{(0)}$ and $\epsilon^{(1)}$. These two odd elements satisfy the graded commutation relations:

$$[f_+, f_-] = -2\epsilon^{(0)}\epsilon^{(1)} E,$$

According to relation (16) the elements $f_\pm \in \mathrm{Ker}(E)$ give rise the symmetry transformations $\delta_{f_\pm}$. These transformations satisfy the basic $N = 2$ supersymmetry relations

$$[\delta_{f_+}, \delta_{f_-}] A_0 = -2\epsilon^{(0)}\epsilon^{(1)} \partial_x A_0, \qquad [\delta_{f_\pm}, \delta_{f_\pm}] A_0 = 0\,.$$

For the particular case of $p = 1$, i.e. $\mathrm{sl}(2,1)$, we find the supersymmetry transformations to be

$$\begin{aligned}
\delta_{f_\pm} b_1 &= \pm\epsilon^{(1)} \left(\partial_x F_1 - b_1 \int \bar{b}_1 F_1 + F_1 \int (b_1 \bar{b}_1 + F_1 \bar{F}_1)\right), \\
\delta_{f_\pm} \bar{b}_1 &= -\epsilon^{(0)} \bar{F}_1 \pm \epsilon^{(1)} \bar{b}_1 \int \bar{b}_1 F_1, \\
\delta_{f_\pm} \bar{F}_1 &= \pm\epsilon^{(1)} \left(\partial_x \bar{b}_1 - \bar{b}_1 \int (b_1 \bar{b}_1 + F_1 \bar{F}_1) + \bar{F}_1 \int \bar{b}_1 F_1\right), \\
\delta_{f_\pm} F_1 &= \epsilon^{(0)} b_1 \mp \epsilon^{(1)} F_1 \int \bar{b}_1 F_1,
\end{aligned} \tag{21}$$

These are indeed symmetries of the equations of motion (10). For the $\widehat{\mathrm{sl}}(p+1,p)$ case, the supersymmetry transformation generated by $f_\pm$ in (20) are explicitly listed in the appendix A.

Let us now calculate the Hamiltonian densities. According to [17], the Hamiltonian densities are given by

$$\mathcal{H}_n = -\,\mathrm{tr}\left(E^{(0)} A^{(-n)}\right) = \frac{1}{2} \sum_{k=0}^{n-1} \mathrm{tr}\left(A^{(-k)} A^{(1+k-n)}\right) \,, \tag{22}$$

where $A^{(-n)}$ are defined via

$$\partial_x \Theta \, \Theta^{-1} = \sum_{k=1}^{\infty} A^{(-k)} \lambda^{-k} \, ,$$

where $\partial_x \Theta \, \Theta^{-1}$ is the quantity, which entered eqn. (11). The symbol tr in expression (22) denotes a trace for the super algebra $\mathrm{sl}(p+1,p)$.

Now, we explicitly work out the first few expressions. Inserting $n=1$ in (22) we obtain:

$$\mathcal{H}_1 \;=\; -\,\mathrm{tr}(E^{(0)} A^{(-1)}) = \frac{1}{2}\,\mathrm{tr}(A_0^2) = \sum_{i=1}^{p} \left(b_i \bar{b}_i + F_i \bar{F}_i \right) \tag{23}$$

where A_0 is given by (7). In order to evaluate $\mathcal{H}_2$ we use (13) to obtain

$$\mathcal{H}_2 \;=\; \mathrm{tr}\left(A_0 \partial_x \theta_{\mathcal{I}}^{(-1)} \right) = \sum_{i=1}^{p} \left(\bar{b}_i \partial_x b_i + \bar{F}_i \partial_x F_i \right) \tag{24}$$

For $n=3$, we find

$$\mathcal{H}_3 \;=\; -\frac{1}{2}\,\mathrm{tr}\left(2 A_0 A^{(-2)} + A^{(-1)^2} \right) \tag{25}$$

where

$$A^{(-1)} = -\partial_x \theta^{(-1)}, \quad A^{(-2)} = \partial_x \theta^{(-2)} - \frac{1}{2}[\partial_x \theta^{(-1)}, \theta^{(-1)}] \tag{26}$$

and, therefore,

$$\begin{aligned} \frac{1}{3}\mathcal{H}_3 \;=\; & \sum_{j=1}^{p} \partial_x \bar{b}_j \partial_x b_j + \sum_{a=1}^{p} \partial_x \bar{F}_a \partial_x F_a + \sum_{i,j=1}^{p} \bar{b}_j b_j \bar{b}_i b_i + 2 \sum_{i,b=1}^{p} \bar{b}_i b_i \bar{F}_b F_b \\ + \; & \sum_{a,b=1}^{p} \bar{F}_a F_a \bar{F}_b F_b \, . \end{aligned} \tag{27}$$

3.1 Poisson Structure of the second flow

The equations of motion (10) can be derived from the Lagrangian density :

$$\begin{aligned} \mathcal{L}_{t_2} \;=\; & \frac{1}{2}\sum_{i=1}^{p} \partial_t b_i \bar{b}_i - \frac{1}{2}\sum_{i=1}^{p} b_i \partial_t \bar{b}_i - \frac{1}{2}\sum_{a=1}^{p} \partial_t F_a \bar{F}_a + \frac{1}{2}\sum_{a=1}^{p} F_a \partial_t \bar{F}_a - \sum_{j=1}^{p} \partial_x \bar{b}_j \partial_x b_j \\ - \; & \sum_{a=1}^{p} \partial_x \bar{F}_a \partial_x F_a - \sum_{i,j=1}^{p} \bar{b}_j b_j \bar{b}_i b_i - 2\sum_{i,b=1}^{p} \bar{b}_i b_i \bar{F}_b F_b - \sum_{a,b=1}^{p} \bar{F}_a F_a \bar{F}_b F_b \end{aligned} \tag{28}$$

The canonical momenta are given by

$$\begin{aligned} P_{b_i} = \frac{\delta \mathcal{L}_{t_2}}{\delta \dot{b}_i} = \frac{1}{2}\bar{b}_i, \quad P_{\bar{b}_i} = \frac{\delta \mathcal{L}_{t_2}}{\delta \dot{\bar{b}}_i} = -\frac{1}{2} b_i, \\ P_{F_a} = \frac{\delta \mathcal{L}_{t_2}}{\delta \dot{F}_a} = -\frac{1}{2}\bar{F}_a, \quad P_{\bar{F}_a} = \frac{\delta \mathcal{L}_{t_2}}{\delta \dot{\bar{F}}_a} = -\frac{1}{2} F_a, \end{aligned} \tag{29}$$

which satisfy the usual canonical Poisson brackets

$$\{P_{b_i}(x), b_j(y)\}_P = \{P_{\bar{b}_i}(x), \bar{b}_j(y)\}_P = -\delta_{ij}\delta(x-y),$$
$$\{P_{F_a}(x), F_b(y)\}_P = \{P_{\bar{F}_a}(x), \bar{F}_b(y)\}_P = -\delta_{ab}\delta(x-y)\,.$$

The momenta dependence on the fields in (29) define the primary constraints

$$\Phi_{b_i} = P_{b_i} - \frac{1}{2}\bar{b}_i = 0, \quad \Phi_{\bar{b}_i} = P_{\bar{b}_i} + \frac{1}{2}b_i = 0,$$
$$\Phi_{F_a} = P_{F_a} + \frac{1}{2}\bar{F}_a = 0, \quad \Phi_{\bar{F}_a} = P_{\bar{F}_a} + \frac{1}{2}F_a = 0, \tag{30}$$

and accordingly the Poisson bracket algebra is given in terms of the Dirac brackets

$$\{b_i(x), \bar{b}_j(y)\}_D = -\delta_{ij}\delta(x-y), \quad \{F_a(x), \bar{F}_b(y)\}_D = -\delta_{ab}\delta(x-y)\,, \tag{31}$$

which together with the total Hamiltonian $H_3 = \int dx \mathcal{H}_3$ obtained from (27) reproduce the equations of motion (10) via formula $\partial X/\partial t_2 = \{X, H_3\}_D$.

A second bracket characterized by P_2 can be obtained from the recursive relation

$$P_2 = \mathcal{R}P_1\,, \tag{32}$$

where P_1 is the first bracket defined in (31) and $\mathcal{R}$ denotes the recursion operator relating consecutive time evolutions, i.e.

$$\partial_{t_m}A_0 = [E, \partial_x\partial_{t_{m-1}}A_0] - [A_0, \partial_x^{-1}[A_0, [E, \partial_{t_{m-1}}A_0]]] = \mathcal{R}\partial_{t_{m-1}}A_0\,.$$

In components this recursion relation for flows reads as :

$$\partial_{t_m}\xi_{i,l} = \sum_{k=1}^{4}\sum_{j=1}^{p}\mathcal{R}_{i,k}^{l,j}\partial_{t_{m-1}}\xi_{k,j}\,, \quad i = 1, \ldots, 4, \quad l = 1, \ldots, p \tag{33}$$

where the symbol $\xi_{i,j}$ represents all the fields contained in A_0 and introduced in equation (13) through $\xi_{1,j} = \bar{b}_j$, $\xi_{2,j} = b_j$, $\xi_{3,j} = F_j$ and $\xi_{4,j} = \bar{F}_j$. For the case of sl$(2,1)$ with $p = 1$ one finds the pseudo-differential operator expressions for the recursion operator $\mathcal{R}_{i,k} = \mathcal{R}_{i,k}^{1,1}$:

$$\begin{aligned}
&\mathcal{R}_{11} = \partial_x - 2\bar{b}\partial_x^{-1}b + \bar{F}\partial_x^{-1}F, && \mathcal{R}_{12} = -2\bar{b}\partial_x^{-1}\bar{b},\\
&\mathcal{R}_{13} = \bar{b}\partial_x^{-1}\bar{F} + \bar{F}\partial_x^{-1}\bar{b}, && \mathcal{R}_{14} = -\bar{b}\partial_x^{-1}F,\\
&\mathcal{R}_{21} = 2b\partial_x^{-1}b, && \mathcal{R}_{22} = -\partial_x + 2b\partial_x^{-1}\bar{b} + F\partial_x^{-1}\bar{F},\\
&\mathcal{R}_{23} = b\partial_x^{-1}\bar{F}, && \mathcal{R}_{24} = F\partial_x^{-1}b + b\partial_x^{-1}F,\\
&\mathcal{R}_{31} = F\partial_x^{-1}b + b\partial_x^{-1}F, && \mathcal{R}_{32} = F\partial_x^{-1}\bar{b},\\
&\mathcal{R}_{33} = -\partial_x + b\partial_x^{-1}\bar{b}, && \mathcal{R}_{34} = 0,\\
&\mathcal{R}_{41} = -\bar{F}\partial_x^{-1}b_1, && \mathcal{R}_{42} = -\bar{F}\partial_x^{-1}\bar{b} - \bar{b}\partial_x^{-1}\bar{F},\\
&\mathcal{R}_{43} = 0, && \mathcal{R}_{44} = \partial_x - \bar{b}\partial_x^{-1}b,
\end{aligned}$$

where ∂_x and the pseudo-differential symbol ∂_x^{-1} are acting on all the fields appearing to their right.

In the appendix B we give the general expression for $\mathcal{R}$ in case of arbitrary sl$(p+1,p)$. These expressions allow an easy derivation of the second bracket.

4 Relativistic Model

Consider the negative grade time evolution equations given by

$$\partial_{t_{-j}} A_0 - \partial_x (D^{(-1)} + D^{(-2)} + \cdots + D^{(-j)}) - [A_0 + E, D^{(-1)} + D^{(-2)} + \cdots + D^{(-j)}] = 0 \quad (34)$$

Here, we only consider $j = 1$. We introduce notation $z = t_{-1}$ and $\bar{z} = -x$. Thus,

$$\partial_z = \frac{\partial}{\partial t_{-1}}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = -\partial_x \, ,$$

and the above evolution equation becomes

$$\partial_z A_0 + \bar{\partial} D^{(-1)} - [A_0 + E, D^{(-1)}] = 0 \, ,$$

with the following solution

$$D^{(-1)} = B E^{(-1)} B^{-1}, \quad A_0 = \bar{\partial} B \, B^{-1} \quad (35)$$

where B is the group element of sl$(p+1, p)$ satisfying the Leznov-Saveliev's equations

$$\bar{\partial}(B^{-1} \partial_z B) + [E^{(-1)}, B^{-1} E B] = 0, \qquad \partial_z(\bar{\partial} B \, B^{-1}) + [B E^{(-1)} B^{-1}, E] = 0 \, . \quad (36)$$

It is customary to think about coordinates $z, \bar{z}$ as light-cone coordinates. Such association allows us to interpret the model as being relativistic.

The group element B transforms under the symmetry transformations (15) as

$$\delta_{X_n} B = -\left(\Theta X_n \Theta^{-1}\right)_0 B \, , \quad n \geq 0 \, . \quad (37)$$

For $n = 0$ and $n = 1$ the above relation takes form

$$\delta_{X_0} B = -X_0 \, B, \qquad \delta_{X_1} B = \left[X_1, \theta^{(-1)}\right] B \, . \quad (38)$$

In particular, for $X_1 = E$ the above definition yields

$$\delta_E B = -\left(\Theta E \Theta^{-1}\right)_0 B = -[\theta^{(-1)}, E] B = -A_0 B$$

recalling that $A_0 = -\partial_x B \, B^{-1}$ we obtain $\delta_E B = \partial_x B$, confirming again that E generates the first positive isospectral flow $t_1 = x$.

These transformations leave the above equations of motion (36) invariant. In particular, the equations of motion (36) remain unchanged under the $N = 2$ supersymmetry transformations $\delta_{f_\pm}$.

Notice that eqns. (36) yields chiral currents associated to the Kernel of E, i.e. $\mathcal{K} = \{X \in$ sl$(p+1, p), [X, E] = 0\}$. In order to make connection with the super AKNS model (7), i.e. $A_0 \in \mathcal{I}$ in (35), we impose the following subsidiary constraints

$$\mathrm{tr}\left(X \bar{\partial} B B^{-1}\right) = \mathrm{tr}\left(X B^{-1} \partial_z B\right) = 0, \quad X \in \mathcal{K} \quad (39)$$

4.1 Relativistic Super Lund-Regge Model

In order to define the relativistic member of the so-called super Lund-Regge hierarchy we consider the zero grade group element $B \in G_0 = SL(2,1)$ parametrized as

$$B = e^{\tilde{\chi}E_{-\alpha_1}} e^{\tilde{f}_1 E_{-\alpha_1-\alpha_2}} e^{\tilde{f}_2 E_{\alpha_2}} e^{\frac{1}{2}\varphi_1(\alpha_1+\alpha_2)\cdot H + \frac{1}{2}\varphi_2(\alpha_2)\cdot H} e^{\tilde{g}_2 E_{-\alpha_2}} e^{\tilde{g}_1 E_{\alpha_1+\alpha_2}} e^{\tilde{\psi}E_{\alpha_1}} \tag{40}$$

where we are using the basis $E_{\pm\alpha_1}, E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}, \alpha_1 \cdot H$ and $\alpha_2 \cdot H$. Changing to the natural variables

$$\begin{array}{lll} \tilde{\psi} = \psi e^{-\frac{\varphi_1+\varphi_2}{2}}, & \tilde{g_1} = g_1 e^{-\frac{\varphi_2}{2}}, & \tilde{f_1} = f_1 e^{-\frac{\varphi_2}{2}} \\ \tilde{\chi} = \chi e^{-\frac{\varphi_1+\varphi_2}{2}}, & \tilde{g_2} = g_2 e^{-\frac{\varphi_1}{2}}, & \tilde{f_2} = f_2 e^{-\frac{\varphi_1}{2}} \end{array} \tag{41}$$

we find for the currents

$$\begin{aligned} J &= B^{-1}\partial_z B = J_{-\alpha_1}E_{\alpha_1} + J_{-\alpha_2}E_{\alpha_2} + J_{-\alpha_1-\alpha_2}E_{\alpha_1+\alpha_2} + J_{(\alpha_1+\alpha_2)\cdot H}\,(\alpha_1+\alpha_2)\cdot H \\ &- J_{\alpha_2\cdot H}\,\alpha_2\cdot H + J_{\alpha_1}E_{-\alpha_1} + J_{\alpha_2}E_{-\alpha_2} + J_{\alpha_1+\alpha_2}E_{-(\alpha_1+\alpha_2)}, \\ \bar{J} &= \bar{\partial}BB^{-1} = \bar{J}_{-\alpha_1}E_{\alpha_1} + \bar{J}_{-\alpha_2}E_{\alpha_2} + \bar{J}_{-\alpha_1-\alpha_2}E_{\alpha_1+\alpha_2} + \bar{J}_{(\alpha_1+\alpha_2)\cdot H}\,(\alpha_1+\alpha_2)\cdot H \\ &- \bar{J}_{\alpha_2\cdot H}\,\alpha_2\cdot H + \bar{J}_{\alpha_1}E_{-\alpha_1} + \bar{J}_{\alpha_2}E_{-\alpha_2} + \bar{J}_{\alpha_1+\alpha_2}E_{-(\alpha_1+\alpha_2)} \end{aligned} \tag{42}$$

with the coefficients $J_{-\alpha_1}, \ldots, \bar{J}_{\alpha_1+\alpha_2}$ shown in Appendix D.

The constraints (39)

$$J_{\pm(\alpha_1+\alpha_2)} = \bar{J}_{\pm(\alpha_1+\alpha_2)} = J_{(\alpha_1+\alpha_2)\cdot H} = \bar{J}_{(\alpha_1+\alpha_2)\cdot H} = J_{\alpha_2\cdot H} = \bar{J}_{\alpha_2\cdot H} = 0 \tag{43}$$

lead to the following equations for the non local fields

$$\begin{aligned} \partial_z f_1 &= \frac{1}{2}f_1\partial_z\varphi_2 + g_2[\partial_z\chi - \frac{1}{2}\chi(\partial_z\varphi_1 + \partial_z\varphi_2)], \\ \partial_z g_1 &= \psi\partial_z f_2 + \frac{1}{2}g_1\partial_z\varphi_2 - \frac{1}{2}\psi f_2\partial_z\varphi_1, \\ \bar{\partial} f_1 &= \chi\bar{\partial}g_2 + \frac{1}{2}f_1\bar{\partial}\varphi_2 - \frac{1}{2}\chi g_2\bar{\partial}\varphi_1, \\ \bar{\partial} g_1 &= \frac{1}{2}g_1\bar{\partial}\varphi_2 + f_2[\bar{\partial}\psi - \frac{1}{2}\psi(\bar{\partial}\varphi_1 + \bar{\partial}\varphi_2)], \\ \partial_z\varphi_1 &= \frac{\psi[\partial_z\chi(1+g_2f_2) + \frac{1}{2}\chi g_2\partial_z f_2]}{1+\psi\chi(1+\frac{5}{4}g_2f_2)}, \\ \partial_z\varphi_2 &= \frac{\psi\partial_z\chi(1+\frac{3}{2}g_2f_2) - g_2\partial_z f_2 - \frac{1}{2}\psi\chi g_2\partial_z f_2}{1+\psi\chi(1+\frac{5}{4}g_2f_2)}, \\ \bar{\partial}\varphi_1 &= \frac{\chi[\bar{\partial}\psi(1+g_2f_2) + \frac{1}{2}\psi\bar{\partial}g_2 f_2]}{1+\psi\chi(1+\frac{5}{4}g_2f_2)}, \\ \bar{\partial}\varphi_2 &= \frac{\chi\bar{\partial}\psi(1+\frac{3}{2}g_2f_2) + (\frac{1}{2}\psi\chi+1)f_2\bar{\partial}g_2}{1+\psi\chi(1+\frac{5}{4}g_2f_2)} \end{aligned} \tag{44}$$

The equations of motion are given in the Leznov-Saveliev form (36), after eliminating the non-local fields with help of equation (44). In components they read

$$
\begin{aligned}
&(1+g_2f_2)[\bar{\partial}\partial_z\chi - \frac{1}{2}\bar{\partial}\chi(\partial_z\varphi_1+\partial_z\varphi_2) - \frac{1}{2}\chi(\bar{\partial}\partial_z\varphi_1+\bar{\partial}\partial_z\varphi_2) + \frac{1}{2}(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2) \\
&\qquad (\partial_z\chi - \frac{1}{2}\chi(\partial_z\varphi_1+\partial_z\varphi_2))] + \bar{\partial}(g_2f_2)[\partial_z\chi - \frac{1}{2}\chi(\partial_z\varphi_1+\partial_z\varphi_2)] + \chi = 0,
\end{aligned} \tag{45}
$$

$$
\bar{\partial}\partial_z g_2 + \frac{1}{2}\bar{\partial}g_2\partial_z\varphi_1 + \frac{1}{2}g_2\bar{\partial}\partial_z\varphi_1 - \frac{1}{2}\partial_z g_2\bar{\partial}\varphi_1 - \frac{1}{4}g_2\bar{\partial}\varphi_1\partial_z\varphi_1 + (1+\psi\chi)g_2 = 0 \tag{46}
$$

$$
\begin{aligned}
&(1+g_2f_2)[\partial_z\bar{\partial}\psi - \frac{1}{2}\partial_z\psi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2) - \frac{1}{2}\psi(\partial_z\bar{\partial}\varphi_1+\partial_z\bar{\partial}\varphi_2) + \frac{1}{2}(\partial_z\varphi_1+\partial_z\varphi_2) \\
&\qquad (\bar{\partial}\psi - \frac{1}{2}\psi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2))] + \partial_z(g_2f_2)[\bar{\partial}\psi - \frac{1}{2}\psi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2)] + \psi = 0
\end{aligned} \tag{47}
$$

$$
\partial_z\bar{\partial}f_2 + \frac{1}{2}\partial_z f_2\bar{\partial}\varphi_1 + \frac{1}{2}f_2\partial_z\bar{\partial}\varphi_1 - \frac{1}{2}\bar{\partial}f_2\partial_z\varphi_1 - \frac{1}{4}f_2\partial_z\varphi_1\bar{\partial}\varphi_1 + (1+\psi\chi)f_2 = 0 \tag{48}
$$

where the non local fields φ_1 and φ_2 are given in equation (44).

4.2 Connection between AKNS and Lund-Regge Variables

On basis of (35) we find the following non-local relations between the AKNS and the relativistic Lund-Regge variables :

$$
\begin{aligned}
\bar{b}_1 = \bar{J}_{-\alpha_1} &= \frac{e^{\frac{1}{2}(\varphi_1+\varphi_2)}}{1+f_2g_2}\left(\bar{\partial}\psi - \frac{1}{2}\psi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2)\right), \\
F_1 = \bar{J}_{-\alpha_2} &= e^{-\frac{1}{2}\varphi_1}\left(\bar{\partial}f_2 + \frac{1}{2}f_2\bar{\partial}\varphi_1\right), \\
b_1 = \bar{J}_{\alpha_1} &= e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\left(\bar{\partial}\chi + \frac{1}{2}\chi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2) - \chi f_2\bar{\partial}g_2 - \frac{1}{2}\chi\bar{\partial}\varphi_1 g_2f_2 + e^{\frac{1}{2}\varphi_1}f_1\bar{J}_{-\alpha_2}\right. \\
&- \left.\chi^2 e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}\right), \\
\bar{F}_1 = \bar{J}_{\alpha_2} &= e^{\frac{1}{2}\varphi_1}\left(\bar{\partial}g_2 - \frac{1}{2}g_2\bar{\partial}\varphi_1 + e^{-\frac{1}{2}(\varphi_1+\varphi_2)}f_1\bar{J}_{-\alpha_1}\right)
\end{aligned} \tag{49}
$$

Notice that they are all non local fields in terms of the Lund-Regge variables due to the presence of φ_1, φ_2 and f_1 defined by (44). Also, we have seen that in terms of the AKNS variables, the models are invariant under supersymmetry transformations (21).

5 Soliton Solutions

5.1 Dressing Transformations and Vertex Operators

The dressing transformation and the vertex operators method represents a powerful tool for the construction of solitons solutions of integrable models. The dressing transformation

relates two solutions of the equations of motion written in the zero curvature representation. In particular, it relates the vacuum and the 1-soliton solutions by a gauge transformation,

$$\mathcal{A}_\mu = \Theta_\pm \mathcal{A}_\mu^{vac} \Theta_\pm^{-1} - (\partial_\mu \Theta_\pm)\, \Theta_\pm^{-1} \tag{50}$$

where

$$\Theta_- = e^{t(-1)+\cdots}, \qquad \Theta_+ = e^{v(0)} e^{v(1)} \cdots \tag{51}$$

The zero curvature representation implies for pure gauge solutions:

$$\mathcal{A}_\mu^{vac} = -\partial_\mu T_0 T_0^{-1}, \qquad \mathcal{A}_\mu = -\partial_\mu T T^{-1} \tag{52}$$

which allows the following relation

$$T = \Theta_\pm T_0, \;\; i.e.\;\; \Theta_+ T = \Theta_- T_0 g\,, \tag{53}$$

where $g \in \hat{G}$ is an arbitrary constant element of the corresponding affine group. Suppose T_0 represents the vacuum solution,

$$T_0 = \exp(-t_n E^{(n)}) \exp(-xE), \qquad n > 1 \tag{54}$$

i.e.,

$$\mathcal{A}_{t_n}^{vac} = E^{(n)}, \qquad \mathcal{A}_x^{vac} = E\,. \tag{55}$$

As consequence of (50) with (55) and (51) we can determine $\Theta_\pm$. Consider for instance eqn. (50) for A_x and Θ_-. It determines its zero grade component $t(-1)$ through

$$A_0 = [t(-1), E] \tag{56}$$

Notice, that if we enlarge the loop algebra by adding central terms, consistency requires the introduction of a additional field ν associated with the extension $A_0 \to A_0 + \partial_x \nu \hat{c}$ in constructing the Lax operator. For the specific sl$(2,1)$ model:

$$A_0 = \bar{b}_1 E_{\alpha_1}^{(0)} + b_1 E_{-\alpha_1}^{(0)} + F_1 E_{\alpha_2}^{(0)} + \bar{F}_1 E_{-\alpha_2}^{(0)} + \partial_x \nu \hat{c} \tag{57}$$

we find

$$t(-1) = -\bar{b}_1 E_{\alpha_1}^{(-1)} + b_1 E_{-\alpha_1}^{(-1)} + F_1 E_{\alpha_2}^{(-1)} - \bar{F}_1 E_{-\alpha_2}^{(-1)} + \partial_x \nu \frac{1}{2}(\alpha_1 + \alpha_2) \cdot H^{(-1)} \tag{58}$$

From (53) we find the solution

$$\begin{aligned}
\tau_0 &\equiv e^{\nu} =< \lambda_0 | T_0 g T_0^{-1} | \lambda_0 >, \\
\tau_1 &\equiv \bar{b}_2 \tau_0 =< \lambda_0 | E_{-\alpha_1}^{(1)} T_0 g T_0^{-1} | \lambda_0 >, \\
\tau_2 &\equiv b_2 \tau_0 = - < \lambda_0 | E_{\alpha_1}^{(1)} T_0 g T_0^{-1} | \lambda_0 >, \\
\tau_3 &\equiv F_1 \tau_0 =< \lambda_0 | E_{-\alpha_2}^{(1)} T_0 g T_0^{-1} | \lambda_0 >, \\
\tau_1 &\equiv \bar{F}_1 \tau_0 =< \lambda_0 | E_{\alpha_1}^{(1)} T_0 g T_0^{-1} | \lambda_0 >,
\end{aligned} \tag{59}$$

For the relativistic model let us consider eqn. (50) for A_x and θ_+. We find

$$A_0 = -\partial_x e^{v(0)} \, e^{-v(0)} \tag{60}$$

Since,for the affine case, $A_0 = -\partial_x BB^{-1} - \partial_x \nu \hat{c}$,

$$e^{v(0)} = Be^{\nu \hat{c}} \tag{61}$$

From (53) we find the solution

$$< \lambda_r | Be^{\nu \hat{c}} | \lambda_t > = < \lambda_r | T_0 g T_0^{-1} | \lambda_t > \tag{62}$$

We now chose $|\lambda_r >$, $|\lambda_t >$ to be the following weight states

$$|\rho > = |\frac{1}{2}\alpha_1 >, \qquad |\epsilon > = | -\frac{1}{2}\alpha_1 - \alpha_2 > \tag{63}$$

and obtain

$$\begin{aligned}
< \rho | Be^{\nu \hat{c}} | \rho > &= \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} = < \rho | T_0 g T_0^{-1} | \rho >, \\
< \epsilon | Be^{\nu \hat{c}} | \epsilon > &= \tau_0 e^{\frac{1}{2}(\varphi_1 - \varphi_2)} = < \epsilon | T_0 g T_0^{-1} | \epsilon >, \\
< \rho | Be^{\nu \hat{c}} E_{-\alpha_1 - \alpha_2} | \rho > &= \frac{1}{2} \tilde{g}_1 \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} = < \rho | T_0 g T_0^{-1} E_{-\alpha_1 - \alpha_2} | \rho >, \\
< \rho | Be^{\nu \hat{c}} E_{\alpha_2} | \rho > &= -\frac{1}{2} \tilde{g}_2 \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} = < \rho | T_0 g T_0^{-1} E_{\alpha_2} | \rho >, \\
< \rho | Be^{\nu \hat{c}} E_{-\alpha_1} | \rho > &= \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} \left(\tilde{\psi} + \frac{1}{2} \tilde{g}_2 \tilde{g}_1 \right) = < \rho | T_0 g T_0^{-1} E_{-\alpha_1} | \rho >, \\
\rho | E_{\alpha_1 + \alpha_2} Be^{\nu \hat{c}} | \rho > &= \frac{1}{2} \tilde{f}_1 \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} = < \rho | E_{\alpha_1 + \alpha_2} T_0 g T_0^{-1} | \rho >, \\
< \rho | E_{-\alpha_2} Be^{\nu \hat{c}} | \rho > &= -\frac{1}{2} \tilde{f}_2 \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} = < \rho | E_{-\alpha_2} T_0 g T_0^{-1} | \rho >, \\
< \rho | E_{\alpha_1} Be^{\nu \hat{c}} | \rho > &= \tau_0 e^{\frac{1}{2}(\varphi_1 + \varphi_2)} \left(\tilde{\psi} + \frac{1}{2} \tilde{g}_2 \tilde{g}_1 \right) = < \rho | E_{\alpha_1} T_0 g T_0^{-1} | \rho >,
\end{aligned} \tag{64}$$

5.2 Vertex Operators and Solutions

In order to evaluate the matrix elements in the r.h.s. of eqns. (59) and (64) it is instructive to introduce the eigenvectors $(\mathcal{F}(\gamma))$ of E^m, i.e.,

$$[E^{(m)}, \mathcal{F}_i(\gamma)] = f_i^m(\gamma) \mathcal{F}_i(\gamma), \tag{65}$$

such that we can classify the soliton solutions in terms of the constant group element $g = e^{\mathcal{F}_1(\gamma_1)} e^{\mathcal{F}_2(\gamma_2)} \cdots$ and hence

$$T_0 g T_0^{-1} = \exp\left(\rho_1(\gamma_1) \mathcal{F}_1(\gamma_1) + \rho_2(\gamma_2) \mathcal{F}_2(\gamma_2) + \cdots \right) \tag{66}$$

where $\rho_i(\gamma_i) = \exp\left(-t_n f_i^n - x f_i^1\right)$.

For $\mathcal{G} = \mathrm{sl}(2,1)$ model, $E^{(1)} = (\alpha_1 + \alpha_2)\cdot H^{(1)}$ and its eigenstates are given by

$$
\begin{aligned}
\mathcal{F}_1(\gamma) &= \sum_{n\in\mathbb{Z}} \left(E^{(n)}_{\alpha_1} + a_1 E^{(n)}_{-\alpha_2}\right)\gamma^{-n},\\
\mathcal{F}_2(\gamma) &= \sum_{n\in\mathbb{Z}} \left(E^{(n)}_{-\alpha_1} + a_2 E^{(n)}_{\alpha_2}\right)\gamma^{-n},
\end{aligned}
\tag{67}
$$

with eigenvalues $f_1^m(\gamma) = \gamma^m, \quad f_2^m(\gamma) = -\gamma^{-m}$. We now consider soliton solutions associated to $g = \exp(\mathcal{F}_1(\gamma_1))\exp(\mathcal{F}_2(\gamma_2))$ and

$$
T_0 g T_0^{-1} = \left(1 + \rho_1(\gamma_1)\mathcal{F}_1(\gamma_1)\right)\left(1 + \rho_2(\gamma_2)\mathcal{F}_2(\gamma_2)\right) \tag{68}
$$

leading to the following matrix elements

$$
\begin{aligned}
\tau_0 &= <0|T_0 g T_0^{-1}|0> = 1 + \Gamma \rho_1' \rho_2',\\
\tau_1 &= <0|E^{(1)}_{-\alpha_1} T_0 g T_0^{-1}|0> = -\gamma_2 \rho_2',\\
\tau_2 &= <0|T_0 g T_0^{-1} E^{(-1)}_{\alpha_1}|0> = \gamma_1 \rho_1',\\
\tau_3 &= <0|E^{(1)}_{-\alpha_2} T_0 g T_0^{-1}|0> = -a_2 \gamma_2 \rho_2',\\
\tau_4 &= <0|T_0 g T_0^{-1} E^{(-1)}_{\alpha_2}|0> = a_1 \gamma_1 \rho_1'
\end{aligned}
\tag{69}
$$

where $b = a_1 a_2$, $\Gamma_0 = \frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}$ and $\Gamma = (1-b)\Gamma_0$. We therefore find the solution for the non relativistic model (84),

$$
\begin{aligned}
\bar{b}_1 &= \frac{\gamma_1 \rho_1'}{1 + \Gamma \rho_1' \rho_2'}, \quad b_1 = -\frac{\gamma_2 \rho_2'}{1 + \Gamma \rho_1' \rho_2'},\\
F_1 &= -a_2 \frac{\gamma_2 \rho_2'}{1 + \Gamma \rho_1' \rho_2'}, \quad \bar{F}_1 = a_1 \frac{\gamma_1 \rho_1'}{1 + \Gamma \rho_1' \rho_2'}
\end{aligned}
\tag{70}
$$

where for the non relativistic model $\rho_1'(\gamma_1) = e^{-t_2 {\gamma_1}^2 - \gamma_1 x}, \quad \rho_2'(\gamma_1) = e^{t_2 {\gamma_2}^2 + \gamma_2 x}$.

For the relativistic model described by equations of motion (45)-(48) we find the following matrix elements

$$
\begin{aligned}
<\rho|T_0 g T_0^{-1}|\rho> &= 1 + \rho_1\rho_2 \frac{\gamma_1}{(\gamma_1-\gamma_2)^2}\left(\gamma_1 - \frac{1}{2}(\gamma_1+\gamma_2)b\right),\\
<\epsilon|T_0 g T_0^{-1}|\epsilon> &= 1 + \rho_1\rho_2 \frac{\gamma_1}{(\gamma_1-\gamma_2)^2}\left(\gamma_1 + \frac{1}{2}(\gamma_1-3\gamma_2)b\right),\\
<\rho|T_0 g T_0^{-1} E^{(0)}_{-\alpha_1}|\rho> &= \rho_1,\\
<\rho|E^{(0)}_{\alpha_1} T_0 g T_0^{-1}|\rho> &= \rho_2,\\
<\rho|T_0 g T_0^{-1} E^{(0)}_{\alpha_2}|\rho> &= -\frac{1}{2} a_1 \rho_1,\\
<\rho|E^{(0)}_{-\alpha_2} T_0 g T_0^{-1}|\rho> &= -\frac{1}{2} a_2 \rho_2,\\
<\rho|T_0 g T_0^{-1} E^{(0)}_{-\alpha_1-\alpha_2}|\rho> &= \frac{1}{2} a_1 \rho_1 \rho_2 \frac{\gamma_1}{(\gamma_1-\gamma_2)},\\
<\rho|E^{(0)}_{\alpha_1+\alpha_2} T_0 g T_0^{-1}|\rho> &= \frac{1}{2} a_1 \rho_2 \frac{\gamma_1}{(\gamma_1-\gamma_2)},
\end{aligned}
\tag{71}
$$

which allow us to determine the solution

$$
\begin{aligned}
e^{\frac{1}{2}(\varphi_1+\varphi_2)} &= \frac{1+\frac{\gamma_1}{\gamma_2}\Gamma_0\left(1-b\frac{(\gamma_1+\gamma_2)}{2\gamma_1}\right)\rho_1\rho_2}{\tau_0}, \\
e^{\frac{1}{2}(\varphi_1-\varphi_2)} &= \frac{1+\Gamma_0\left(1+b\frac{(\gamma_1-3\gamma_2)}{2\gamma_2}\right)\rho_1\rho_2}{\tau_0} = 1-\frac{b}{2}\frac{(\gamma_2-\gamma_1)\Gamma_0\rho_1\rho_2}{\gamma_2(1+\Gamma_0\rho_1\rho_2)}, \\
\psi &= \frac{\rho_1}{\tau_0}\left(1-\frac{b\gamma_1\rho_1\rho_2}{2(\gamma_1-\gamma_2)(1+\frac{\gamma_1}{\gamma_2}\Gamma_0\rho_1\rho_2)}\right), \\
\chi &= \frac{\rho_2}{\tau_0}\left(1-\frac{b\gamma_1\rho_1\rho_2}{2(\gamma_1-\gamma_2)(1+\frac{\gamma_1}{\gamma_2}\Gamma_0\rho_1\rho_2)}\right), \\
g_1 &= a_2\frac{\gamma_1\rho_1\rho_2}{(\gamma_1-\gamma_2)\tau_0}e^{-\frac{1}{2}\varphi_1}, \qquad f_1 = a_1\frac{\gamma_1\rho_1\rho_2}{(\gamma_1-\gamma_2)\tau_0}e^{-\frac{1}{2}\varphi_1}, \\
g_2 &= a_1\frac{\rho_1}{\tau_0}e^{-\frac{1}{2}\varphi_2}, \qquad f_2 = a_2\frac{\rho_2}{\tau_0}e^{-\frac{1}{2}\varphi_2},
\end{aligned}
\tag{72}
$$

From the solutions given in (72) it is easy to verify the consistency of the constraints (39) (or (44)). Let us take for instance

$$
\bar{\partial}f_1 = \chi\bar{\partial}g_2 + \frac{1}{2}f_1\bar{\partial}\varphi_2 - \frac{1}{2}g_2\chi\bar{\partial}\varphi_1 \tag{73}
$$

Substituting (72) in both sides of eqn. (73) and identifying $t_{-1} = z, \quad x = -\bar{z}$ using the second eqn. (72) together with the fact that $a_1 b = a_2 b = 0$ we find that the constraint (73) is satisfied. The same can be shown for all eqns. (44) (i.e. all constraints (39) are satisfied).

Finally we verify the connection between the non-relativistic and the relativistic variables given by eqns. (49). Substituting the solutions (72) in (49) and comparing with (70) we conclude that they agree if we associate

$$
\gamma^2 t_2 \to \frac{z}{\gamma} = \frac{t_{-1}}{\gamma}, \qquad x \to -\bar{z} \tag{74}
$$

which implies $\rho(\gamma) \to \rho'(\gamma)$.

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A Supersymmetry transformation for $\widehat{\mathrm{sl}}(p+1,p)$

Let $f_\pm$ be the supersymmetry generators from equation (20). Using the symmetry transformations given in equations (18) and (19) and

$$\begin{aligned} \theta_I^{(-1)} &= \sum_{i=1}^{p}\left(-\bar{b}_i E^{(-1)}_{(e_i-e_{p+1})} + b_i E^{(-1)}_{-(e_i-e_{p+1})}\right) \\ &+ \sum_{a=1}^{p}\left(-\bar{F}_a E^{(-1)}_{(f_a-e_{p+1})} + F_a E^{(-1)}_{-(f_a-e_{p+1})}\right) \end{aligned} \tag{75}$$

and

$$\begin{aligned} \partial_x\theta_K^{(-1)} &= -\sum_{j=1,\, j\neq i}^{p}\sum_{i=1}^{p}\bar{b}_j b_i E^{(-1)}_{(e_i-e_j)} - \sum_{j=1}^{p}\bar{b}_j b_j (e_j-e_{p+1})\cdot H^{(-1)} \\ &- \sum_{j=1}^{p}\sum_{a=1}^{p}\bar{b}_j F_a E^{(-1)}_{(e_i-f_a)} - \sum_{j=1}^{p}\sum_{a=1}^{p} b_j \bar{F}_a E^{(-1)}_{-(e_i-f_a)} \\ &- \sum_{a=1,\, a\neq b}^{p}\sum_{b=1}^{p}\bar{F}_a F_b E^{(-1)}_{(f_a-f_b)} + \sum_{a=1}^{p}\bar{F}_a F_a (f_a-e_{p+1})\cdot H^{(-1)} \end{aligned} \tag{76}$$

obtained from equation (12) we find

$$\begin{aligned} \delta_{f_\pm} b_r &= \pm\left(\partial_x F_r - \sum_{j=1}^{p}\left(b_j\int \bar{b}_j F_r - F_j\int(b_r\bar{b}_j + F_r\bar{F}_j)\right)\right)\epsilon^{(1)}, \\ \delta_{f_\pm}\bar{b}_r &= -\bar{F}_r\epsilon^{(0)} \pm \left(\sum_{j=1}^{p}\bar{b}_j\int\bar{b}_r F_j\right)\epsilon^{(1)}, \\ \delta_{f_\pm}\bar{F}_r &= \pm\left(\partial_x\bar{b}_r - \left(\sum_{j=1}^{p}\bar{b}_j\int(b_j\bar{b}_r + F_j\bar{F}_r)\right) + \bar{F}_r\int\bar{b}_r F_r\right)\epsilon^{(1)}, \\ \delta_{f_\pm} F_r &= b_r\epsilon^{(0)} \mp \sum_{j=1}^{p} F_j\int\bar{b}_j F_r\epsilon^{(1)} \end{aligned} \tag{77}$$

B Recursion Operator for $\widehat{\mathrm{sl}}(p+1,p)$

Using the same notation as in equation (33) we write down expression for the recursion pseudo-differential operator $\mathcal{R}^{lj}_{i,k}$ valid for a general case of $\widehat{\mathrm{sl}}(p+1,p)$. Here, the $l,j = 1,\ldots,p$

and indices $i, k = 1, 2, 3, 4$ label the four modes $\bar{b}_j, b_j, F_j, \bar{F}_j$.

$$
\begin{aligned}
\mathcal{R}^{lj}_{11} &= \delta_{lj}\left(\partial_x - \sum_k \bar{b}_k \partial_x^{-1} b_k + \sum_b \bar{F}_b \partial_x^{-1} F_b\right) - \bar{b}_l \partial_x^{-1} b_j, \\
\mathcal{R}^{lj}_{12} &= -\bar{b}_j \partial_x^{-1} \bar{b}_l - \bar{b}_l \partial_x^{-1} \bar{b}_j, \\
\mathcal{R}^{lj}_{13} &= \bar{b}_l \partial_x^{-1} \bar{F}_j + \bar{F}_j \partial_x^{-1} \bar{b}_l, \\
\mathcal{R}^{lj}_{14} &= -\bar{b}_l \partial_x^{-1} F_j, \\
\mathcal{R}^{lj}_{21} &= b_j \partial_x^{-1} b_l + b_l \partial_x^{-1} b_j, \\
\mathcal{R}^{lj}_{22} &= \delta_{lj}\left(-\partial_x + \sum_k b_k \partial_x^{-1} \bar{b}_k + \sum_b F_b \partial_x^{-1} \bar{F}_b\right) + b_l \partial_x^{-1} \bar{b}_j, \\
\mathcal{R}^{lj}_{23} &= b_l \partial_x^{-1} \bar{F}_j, \\
\mathcal{R}^{lj}_{24} &= F_j \partial_x^{-1} b_l + b_l \partial_x^{-1} F_j, \\
\mathcal{R}^{lj}_{31} &= F_l \partial_x^{-1} b_j + b_j \partial_x^{-1} F_l, \\
\mathcal{R}^{lj}_{32} &= F_l \partial_x^{-1} \bar{b}_j, \\
\mathcal{R}^{lj}_{33} &= \delta_{lj}\left(-\partial_x + \sum_k b_k \partial_x^{-1} \bar{b}_k + \sum_b F_b \partial_x^{-1} \bar{F}_b\right) - F_l \partial_x^{-1} \bar{F}_j \\
\mathcal{R}^{lj}_{34} &= F_l \partial_x^{-1} F_j - F_j \partial_x^{-1} F_l, \\
\mathcal{R}^{lj}_{41} &= -\partial_x^{-1} b_j \bar{F}_l, \\
\mathcal{R}^{lj}_{42} &= -\bar{F}_l \partial_x^{-1} \bar{b}_j - \bar{b}_j \partial_x^{-1} \bar{F}_l \cdot), \\
\mathcal{R}^{lj}_{43} &= -\bar{F}_j \partial_x^{-1} \bar{F}_l + \bar{F}_l \partial_x^{-1} \bar{F}_j \\
\mathcal{R}^{lj}_{44} &= \delta_{lj}\left(\partial_x - \sum_k \bar{b}_k \partial_x^{-1} b_k + \sum_b \bar{F}_b \partial_x^{-1} F_b\right) - \bar{F}_l \partial_x^{-1} F_j.
\end{aligned}
$$

C Case of $\widehat{\mathrm{sl}}(3,2)$

Consider the $\widehat{\mathrm{sl}}(3,2)$ case where

$$
\begin{aligned}
A_0 &= b_1 E^{(0)}_{-\alpha_1-\alpha_2} + \bar{b}_1 E^{(0)}_{\alpha_1+\alpha_2} + b_2 E^{(0)}_{-\alpha_2} + \bar{b}_2 E^{(0)}_{\alpha_2} \\
&+ F_1 E^{(0)}_{\alpha_3} + \bar{F}_1 E^{(0)}_{-\alpha_3} + F_2 E^{(0)}_{\alpha_3+\alpha_4} + \bar{F}_2 E^{(0)}_{-\alpha_3-\alpha_4}
\end{aligned}
\tag{78}
$$

and propose the zero curvature representation

$$
\partial_{t_2} A_0 - \partial_x \left(D^{(0)} + D^{(1)} + D^{(2)}\right) - [A_0 + E, D^{(0)} + D^{(1)} + D^{(2)}] = 0 \tag{79}
$$

from where we find the following solutions for equations (9),

$$\begin{aligned}
D^{(2)} &= (D^{(2)})_{Ker} = (e_1+e_2-f_1-f_2)\cdot H^{(2)}, \\
D^{(1)} &= (D^{(1)})_{IM} = b_1 E^{(1)}_{-\alpha_1-\alpha_2} + b_2 E^{(1)}_{-\alpha_2} + \bar{b}_1 E^{(1)}_{\alpha_1+\alpha_2} + \bar{b}_2 E^{(1)}_{\alpha_2} \\
&\quad + F_1 E^{(1)}_{\alpha_3} + F_2 E^{(1)}_{\alpha_3+\alpha_4} + \bar{F}_1 E^{(1)}_{-\alpha_3} + \bar{F}_2 E^{(1)}_{-\alpha_3-\alpha_4} \\
(D^{(0)})_{IM} &= -\partial_x b_1 E^{(0)}_{-\alpha_1-\alpha_2} - \partial_x b_2 E^{(0)}_{-\alpha_2} + \partial_x \bar{b}_1 E^{(0)}_{\alpha_1+\alpha_2} + \partial_x \bar{b}_2 E^{(0)}_{\alpha_2} \\
&\quad - \partial_x F_1 E^{(0)}_{\alpha_3} - \partial_x F_2 E^{(0)}_{\alpha_3+\alpha_4} + \partial_x \bar{F}_1 E^{(0)}_{-\alpha_3} + \partial_x \bar{F}_2 E^{(0)}_{-\alpha_3-\alpha_4} \\
(D^{(0)})_{Ker} &= -(b_1\bar{b}_1)h^{(0)}_1 - (b_1\bar{b}_1 + b_2\bar{b}_2)h^{(0)}_2 + (F_1\bar{F}_1 + F_2\bar{F}_2)h^{(0)}_3 + (F_2\bar{F}_2)h^{(0)}_4 - (b_1\bar{b}_2)E^{(0)}_{-\alpha_1} \\
&\quad - (b_1\bar{F}_1)E^{(0)}_{-\alpha_1-\alpha_2-\alpha_3} - (b_1\bar{F}_2)E^{(0)}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} - (b_2\bar{b}_1)E^{(0)}_{\alpha_1} - (b_2\bar{F}_1)E^{(0)}_{-\alpha_2-\alpha_3} \\
&\quad - (b_2\bar{F}_2)E^{(0)}_{-\alpha_2-\alpha_3-\alpha_4} - (\bar{b}_1 F_1)E^{(0)}_{\alpha_1+\alpha_2+\alpha_3} - (\bar{b}_1 F_2)E^{(0)}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - (\bar{b}_2 F_1)E^{(0)}_{\alpha_1+\alpha_2} \\
&\quad - (\bar{b}_2 F_2)E^{(0)}_{\alpha_2+\alpha_3+\alpha_4} - (F_1\bar{F}_2)E^{(0)}_{-\alpha_4} - (F_2\bar{F}_1)E^{(0)}_{\alpha_4}
\end{aligned} \tag{80}$$

and the equations of motion

$$\begin{aligned}
\partial_{t_2} b_1 + \partial_x^2 b_1 - 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1 + F_2\bar{F}_2\right) b_1 = 0 \\
\partial_{t_2} b_2 + \partial_x^2 b_2 - 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1 + F_2\bar{F}_2\right) b_2 = 0 \\
\partial_{t_2} \bar{b}_1 - \partial_x^2 \bar{b}_1 + 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1 + F_2\bar{F}_2\right) \bar{b}_1 = 0 \\
\partial_{t_2} \bar{b}_2 - \partial_x^2 \bar{b}_2 + 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1 + F_2\bar{F}_2\right) \bar{b}_2 = 0 \\
\partial_{t_2} F_1 + \partial_x^2 F_1 - 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_2\bar{F}_2\right) F_1 = 0 \\
\partial_{t_2} F_2 + \partial_x^2 F_2 - 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1\right) F_2 = 0 \\
\partial_{t_2} \bar{F}_1 - \partial_x^2 \bar{F}_1 + 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_2\bar{F}_2\right) \bar{F}_1 = 0 \\
\partial_{t_2} \bar{F}_2 - \partial_x^2 \bar{F}_2 + 2\left(b_1\bar{b}_1 + b_2\bar{b}_2 + F_1\bar{F}_1\right) \bar{F}_2 = 0
\end{aligned} \tag{81}$$

Consider

$$\begin{aligned}
X_0 &= \epsilon_{-\alpha_2-\alpha_3} E^{(0)}_{\alpha_2+\alpha_3} + \epsilon_{\alpha_2+\alpha_3} E^{(0)}_{-\alpha_2-\alpha_3} + \epsilon_{-\alpha_2-\alpha_3-\alpha_4} E^{(0)}_{\alpha_2+\alpha_3+\alpha_4} \\
&+ \epsilon_{\alpha_2+\alpha_3+\alpha_4} E^{(0)}_{-\alpha_2-\alpha_3-\alpha_4} + \epsilon_{-\alpha_1-\alpha_2-\alpha_3} E^{(0)}_{\alpha_1+\alpha_2+\alpha_3} + \epsilon_{\alpha_1+\alpha_2+\alpha_3} E^{(0)}_{-\alpha_1-\alpha_2-\alpha_3} \\
&+ \epsilon_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} E^{(0)}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} + \epsilon_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} E^{(0)}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}
\end{aligned} \tag{82}$$

leading from (18) to the supersymmetry transformations

$$\begin{aligned}
\delta_{X_0} b_1 &= \epsilon_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} F_2 + \epsilon_{\alpha_1+\alpha_2+\alpha_3} F_1, \\
\delta_{X_0} b_2 &= \epsilon_{\alpha_2+\alpha_3+\alpha_4} F_2 + \epsilon_{\alpha_2+\alpha_3} F_1, \\
\delta_{X_0} \bar{b}_1 &= \epsilon_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \bar{F}_2 + \epsilon_{-\alpha_1-\alpha_2-\alpha_3} \bar{F}_1, \\
\delta_{X_0} \bar{b}_2 &= \epsilon_{-\alpha_2-\alpha_3-\alpha_4} \bar{F}_2 + \epsilon_{-\alpha_2-\alpha_3} \bar{F}_1, \\
\delta_{X_0} F_1 &= -\epsilon_{-\alpha_2-\alpha_3} b_2 - \epsilon_{-\alpha_1-\alpha_2-\alpha_3} b_1, \\
\delta_{X_0} F_2 &= -\epsilon_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} b_1 - \epsilon_{-\alpha_2-\alpha_3-\alpha_4} b_2, \\
\delta_{X_0} \bar{F}_1 &= \epsilon_{\alpha_2+\alpha_3} \bar{b}_2 + \epsilon_{\alpha_1+\alpha_2+\alpha_3} \bar{b}_1, \\
\delta_{X_0} \bar{F}_2 &= \epsilon_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \bar{b}_1 + \epsilon_{\alpha_2+\alpha_3+\alpha_4} \bar{b}_2,
\end{aligned} \tag{83}$$

Of course when $b_1 = \bar{b}_1 = 0$ and $F_2 = \bar{F}_2 = 0$ we recover the $p = 1$ or $SL(2,1)$ case proposed in ref. [14] corresponding to $A_0 = \bar{b}_2 E_{\alpha_1} + b_2 E_{-\alpha_1} + F_1 E_{\alpha_2} + \bar{F}_1 E_{-\alpha_2}$, i.e.,

$$
\begin{aligned}
\partial_{t_2} b_2 + \partial_x^2 b_2 - 2\left(b_2\bar{b}_2 + F_1\bar{F}_1\right) b_2 = 0 \\
\partial_{t_2} \bar{b}_2 - \partial_x^2 \bar{b}_2 + 2\left(b_2\bar{b}_2 + F_1\bar{F}_1\right) \bar{b}_2 = 0 \\
\partial_{t_2} F_1 + \partial_x^2 F_1 - 2 b_2\bar{b}_2 F_1 = 0 \\
\partial_{t_2} \bar{F}_1 - \partial_x^2 \bar{F}_1 + 2 b_2\bar{b}_2 \bar{F}_1 = 0
\end{aligned}
\tag{84}
$$

D SL(2,1) super-currents

The components of the sl$(2,1)$ currents $J = B^{-1}\partial B$ and $\bar{J} = \bar{\partial} B B^{-1}$ given in equation (42) read in terms of the fields defined in (40) and (41):

$$
\begin{aligned}
J_{\alpha_1} &= e^{\frac{1}{2}(\varphi_1+\varphi_2)}\left(\partial_z\chi - \frac{1}{2}\chi(\partial_z\varphi_1 + \partial_z\varphi_2) + \partial_z f_1 f_2 - \frac{1}{2} f_1 f_2 \partial_z\varphi_2\right), \\
J_{\alpha_2} &= e^{-\frac{1}{2}\varphi_1}\left(\partial g_2 + \frac{1}{2} g_2\partial\varphi_1 + \partial f_1\psi - \frac{1}{2} f_1\partial\varphi_2\psi - \psi g_2 e^{-\frac{1}{2}(\varphi_1+\varphi_2)} J_{\alpha_1}\right), \\
J_{\alpha_1+\alpha_2} &= e^{\frac{1}{2}\varphi_2}\left(\partial f_1 - \frac{1}{2} f_1\partial\varphi_2 - g_2 e^{-\frac{1}{2}(\varphi_1+\varphi_2)} J_{\alpha_1}\right), \\
J_{\alpha_2\cdot H} &= \partial\varphi_2 - \partial f_2 g_2 + \frac{1}{2} f_2 g_2 \partial\varphi_1 - \psi e^{-\frac{1}{2}(\varphi_1+\varphi_2)} J_{\alpha_1}, \\
J_{(\alpha_1+\alpha_2)\cdot H} &= \partial\varphi_1 + \partial f_1 g_1 - \frac{1}{2} f_1 g_1 \partial\varphi_2 - (\psi + g_2 g_1) e^{-\frac{1}{2}(\varphi_1+\varphi_2)} J_{\alpha_1}, \\
J_{-\alpha_1-\alpha_2} &= e^{\frac{1}{2}\varphi_2}\left(\partial g_1 - \frac{1}{2} g_1\partial\varphi_2 - \psi\partial f_2 + \frac{1}{2}\psi\partial\varphi_1 f_2 + g_1 J_{\alpha_2\cdot H}\right), \\
J_{-\alpha_1} &= e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\left(\partial\psi + \frac{1}{2}\psi(\partial\varphi_1 + \partial\varphi_2) + \psi g_2\partial f_2 - \frac{1}{2}\psi\partial\varphi_1 g_2 f_2 + g_1 e^{\frac{1}{2}\varphi_1} J_{\alpha_2}\right. \\
&- \left. \psi^2 e^{-\frac{1}{2}(\varphi_1+\varphi_2)} J_{\alpha_1}\right), \\
J_{-\alpha_2} &= e^{\frac{1}{2}(\varphi_1}\left(\partial f_2 - \frac{1}{2} f_2\partial\varphi_1 + g_1 e^{\frac{1}{2}\varphi_1+\varphi_2)} J_{\alpha_1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{J}_{-\alpha_1} &= e^{\frac{1}{2}(\varphi_1+\varphi_2)}\left(\bar{\partial}\psi - \frac{1}{2}\psi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2) + g_2\bar{\partial}g_1 - \frac{1}{2}g_2g_1\partial_z\varphi_2\right), \\
\bar{J}_{-\alpha_2} &= e^{-\frac{1}{2}\varphi_1}\left(\bar{\partial}f_2 + \frac{1}{2}f_2\bar{\partial}\varphi_1 + \bar{\partial}g_1\chi - \frac{1}{2}g_1\bar{\partial}\varphi_2\chi - \chi f_2 e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}\right), \\
\bar{J}_{-\alpha_1-\alpha_2} &= e^{\frac{1}{2}\varphi_2}\left(\bar{\partial}g_1 - \frac{1}{2}g_1\bar{\partial}\varphi_2 - f_2e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}\right), \\
\bar{J}_{\alpha_2\cdot H} &= \bar{\partial}\varphi_2 + \bar{\partial}g_2f_2 + \frac{1}{2}f_2g_2\bar{\partial}\varphi_1 - \chi e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}, \\
\bar{J}_{(\alpha_1+\alpha_2)\cdot H} &= \bar{\partial}\varphi_1 - \bar{\partial}g_1f_1 - \frac{1}{2}f_1g_1\bar{\partial}\varphi_2 - (\chi + f_1f_2)e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}, \\
\bar{J}_{\alpha_1+\alpha_2} &= e^{-\frac{1}{2}\varphi_2}\left(\bar{\partial}f_1 - \frac{1}{2}f_1\bar{\partial}\varphi_2 - \chi\bar{\partial}g_2 + \frac{1}{2}\chi\bar{\partial}\varphi_1g_2 + f_1\bar{J}_{\alpha_2\cdot H}\right), \\
\bar{J}_{\alpha_1} &= e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\left(\bar{\partial}\chi + \frac{1}{2}\chi(\bar{\partial}\varphi_1+\bar{\partial}\varphi_2) - \chi f_2\bar{\partial}g_2 - \frac{1}{2}\chi\bar{\partial}\varphi_1g_2f_2 + f_1e^{\frac{1}{2}\varphi_1}\bar{J}_{-\alpha_2}\right. \\
&- \left.\chi^2e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}\right), \\
\bar{J}_{\alpha_2} &= e^{\frac{1}{2}\varphi_1}\left(\bar{\partial}g_2 - \frac{1}{2}g_2\bar{\partial}\varphi_1 + f_1e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\bar{J}_{-\alpha_1}\right),
\end{aligned}
\tag{85}
$$

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